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GENERALISED WITT ALGEBRAS AND IDEALIZERS

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Abstract. Let $\Bbbk$ be an algebraically closed field of characteristic zero, and let Γ be an additive subgroup of $\Bbbk$. Results of Kaplansky-Santharoubane and Su classify intermediate series representations of the *generalised Witt algebra* W_Γ in terms of three families, one parameterised by $\mathbb{A}^2$ and two by $\mathbb{P}^1$. In this note, we use the first family to construct a homomorphism Φ from the enveloping algebra $U(W_\Gamma)$ to a skew extension $\Bbbk[\mathbb{A}^2] \rtimes \Gamma$ of the coordinate ring of $\mathbb{A}^2$. We show that the image of Φ is contained in a (double) idealizer subring of this skew extension and that the representation theory of idealizers explains the three families. We further show that the image of $U(W_\Gamma)$ under Φ is not left or right noetherian, giving a new proof that $U(W_\Gamma)$ is not noetherian.

We construct Φ as an application of a general technique to create ring homomorphisms from shift-invariant families of modules. Let G be an arbitrary group and let A be a G-graded ring. A graded A-module M is an *intermediate series* module if M_g is one-dimensional for all $g \in G$. Given a shift-invariant family of intermediate series A-modules parametrised by a scheme X, we construct a homomorphism Φ from A to a skew extension of $\Bbbk[X]$. The kernel of Φ consists of those elements which annihilate all modules in X.

1. Introduction

Fix an algebraically closed ground field $\Bbbk$ of characteristic zero, and let Γ be a finitely generated additive subgroup of $\Bbbk$. The *generalised Witt algebra* W_Γ is the Lie algebra generated by elements $e_\gamma : \gamma \in \Gamma$, with $[e_\gamma, e_\delta] = (\delta - \gamma)e_{\delta+\gamma}$. Recall that an *intermediate series representation* of W_Γ is an indecomposable representation all of whose e_0-eigenspaces are 1-dimensional. It is a theorem of Kaplansky and Santharoubane [KS85] (if $\Gamma = \mathbb{Z}$) and of Su [Su94] (for general Γ) that intermediate series representations of W_Γ come in three families (with two modules represented twice): one family parameterised by $\mathbb{A}^2$ and two parameterised by $\mathbb{P}^1$. In this note we use the first family to construct a homomorphism Φ from $U(W_\Gamma)$ to $T = \Bbbk[\mathbb{A}^2] \rtimes \Gamma$, and show that the existence of the other two families is a consequence of the fact that the image of $U(W_\Gamma)$ is a sub-idealizer in T. We further use the homomorphism Φ to give a new proof that the enveloping algebra of $U(W_\Gamma)$ is not noetherian, a fact originally proved in [SW14].

Since our main method is to construct and then analyze a homomorphism from $U(W_\Gamma)$ to an idealizer in T, we recall some facts about idealizers. We first define T: as a vector space we write $T = \bigoplus_{\gamma\in\Gamma} \Bbbk[a,b]t^\gamma$, with $t^\gamma t^\delta = t^{\gamma+\delta}$ and $t^\gamma f(a,b) = f(a+\gamma, b)t^\gamma =: f^\gamma t^\gamma$. Note that T is a bimodule over $\Bbbk[a,b]$.

An *intermediate series* module M over a Γ-graded ring is an indecomposable Γ-graded module with each M_γ a one-dimensional vector space. It is a generalisation of a point module over an $\mathbb{N}$-graded ring, which is a cyclic graded module with Hilbert series $1/(1-t)$.

For $p = (\alpha, \beta) \in \mathbb{A}^2$, let $I(p)$ be the ideal $(a-\alpha, b-\beta)$ of $\Bbbk[a,b]$. Let $V(p) = T/I(p)T$. It is easy to see that the $V(p)$ are all of the intermediate series right T-modules; more precisely, the right ideals J of T such that T/J is an intermediate series module are precisely the $I(p)T$. Likewise, the intermediate series left T-modules are the $T/TI(p)$. These families are preserved under degree shifting.

We now consider a subring of T. Fix $p_0 \in \mathbb{A}^2$, and let $S = S(p_0) = \Bbbk \oplus I(p_0)T$. The ring S is an *idealizer* in T: the largest subalgebra of T such the right ideal $I(p_0)T$ becomes a two-sided ideal in S. It is known [Rog04] that the representation theory of idealizers involves blowing up. Here for $p \neq p_0$ we have that $V(p) \cong S/(S \cap I(p)T)$ is an intermediate series right S-module. On the other hand, to define an intermediate series right S-module at p_0, we need to consider a point q *infinitely near* to p_0: that is, an ideal $I(q)$ with $I(p_0)^2 \subseteq I(q) \subseteq I(p_0)$ of $\Bbbk[a,b]$ such that $I(p_0)/I(q)$ is one-dimensional. Such ideals are parameterised by the exceptional $\mathbb{P}^1$ in the blowup $\mathrm{Bl}_{p_0}(\mathbb{A}^2)$; more specifically, we can write

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$I(q) = (y(a-\alpha) - x(b-\beta), (a-\alpha)^2, (a-\alpha)(b-\beta), (b-\beta)^2)$ for some $[x:y] \in \mathbb{P}^1$. For such $I(q)$ we have that $I(p_0) + I(q)T$ is a right ideal of S. Let

$$P(q) = S/(I(p_0) + I(q)T).$$

Then $P(q)$ is a intermediate series right S-module. In fact, we have constructed all right ideals J of S such that S/J is an intermediate series S-module; they are parameterised by $\mathrm{Bl}_{p_0}(\mathbb{A}^2)$ but it is sometimes more convenient to consider them as parameterised by $\mathbb{A}^2 \smallsetminus \{p_0\}$ together with $\mathbb{P}^1$.

Left intermediate series S-modules are also parameterised by $\mathrm{Bl}_{p_0}(\mathbb{A}^2)$. For $p \in \mathbb{A}^2 \smallsetminus \{p_0\}$, the left intermediate series module $T/TI(p)$ is isomorphic to $\Big(I(p_0) \oplus \bigoplus_{0\neq\nu\in\Gamma} \Bbbk[a,b]t^\nu\Big) \,/\, \Big((I(p_0) \cap I(p)) \oplus \bigoplus_{0\neq\nu\in\Gamma} t^\nu I(p)\Big)$. We can extend this construction to a family of modules parameterised by $\mathrm{Bl}_{p_0}(\mathbb{A}^2)$ by adding the $\mathbb{P}^1$ of points q infinitely near to p_0:

$$Q(q) = \frac{I(p_0) \oplus \bigoplus_{0\neq\nu\in\Gamma} \Bbbk[a,b]t^\nu}{I(q) \oplus \bigoplus_{0\neq\nu\in\Gamma} t^\nu I(p_0)}.$$

Consider now right intermediate series modules over the double idealiser

$$R = \Bbbk[a,b] + (I(p_0)T \cap TI(p_1))$$

and assume for simplicity that $p_0, p_1 \in \mathbb{A}^2$ have distinct Γ-orbits. These correspond to points of the double blowup $\mathrm{Bl}_{p_0,p_1}(\mathbb{A}^2)$. More precisely, the $V(p)$ are intermediate series modules for $p \in \mathbb{A}^2 \smallsetminus \{p_0, p_1\}$. From the inclusion $R \subseteq \Bbbk \oplus I(p_0)T$ we obtain a family $P(q)$ parameterised by the $\mathbb{P}^1$ of points infinitely near to p_0. Finally, from the inclusion $R \subseteq \Bbbk \oplus TI(p_1)$ we obtain a family $Q(q)$ of right modules parameterised by the $\mathbb{P}^1$ of points infinitely near to p_1 and constructed similarly to the construction of the left modules $Q(q)$ over S.

Let Γ now be an arbitrary group (more generally, a monoid) and let A be a Γ-graded ring. We give a general result in Theorem 2.2 (respectively, Theorem 2.5) which constructs a ring homomorphism (respectively, an anti-homomorphism) $\Phi : A \to \Bbbk[X] \rtimes \Gamma$, where X is a shift-invariant family of right (respectively, left) intermediate series A-modules; this generalises constructions in [ATV91, RZ08, V96].

When we apply this technique to $U(W_\Gamma)$, we show that the image of Φ is contained in a double idealizer R inside the ring T defined in the second paragraph, and we show in Propositions 3.5, 3.6 that the right intermediate series R-modules constructed above restrict to precisely the intermediate series representations of W_Γ. This gives a unified geometric description of what have until now been seen as three distinct families of representations.

We further show in Proposition 4.3 that the image of $U(W_\Gamma)$ under Φ is neither right nor left noetherian. For $\Gamma = \mathbb{Z}$ this was proved in [SW15] as the main step in proving the non-noetherianity of $U(W)$. It follows that $U(W_\Gamma)$ is neither right or left noetherian; other proofs are given in [SW14, SW15].

The general behaviour of idealizers leads one to expect that at idealizers in T at ideals of points in $\Bbbk[a,b]$ will not be noetherian since no points have dense Γ-orbits; see [Sie11] for a precise statement of a related result for $\mathbb{N}$-graded rings. However, infinite orbits are dense in $\mathbb{A}^1$. Thus one expects that the factors $\Phi(U(W_\Gamma))|_{b=\beta}$, which live on the Γ-invariant line $(b = \beta)$ in $\mathbb{A}^2$, are noetherian for all $\beta \in \Bbbk$, and we also show in Proposition 4.6 that this is indeed the case.

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2. Intermediate series modules and ring homomorphisms

It is well-known that ring homomorphisms can be constructed from shift-invariant families of modules. Let A be a (connected $\mathbb{N}$-) graded ring, generated in degree 1. A *point module* over A is a cyclic graded A-module with Hilbert series $1/(1-t)$. Suppose that (right) A-point modules are parameterised by a projective scheme X. Let the point module corresponding to $x \in X$ be M^x. Then the shift functor $\Psi : M \mapsto M[1]_{\geq 0}$ induces an automorphism σ of X so that $\Psi(M^x) \cong M^{\sigma(x)}$.

The following result goes back to [ATV90] (see also [V96]), although in this form it is due to Rogalski and Zhang.

Theorem 2.1. ([RZ08, Theorem 4.4]) *There is an invertible sheaf $\mathcal{L}$ on X so that there is a homomorphism $\phi : A \to B(X, \mathcal{L}, \sigma)$ of graded rings, where $B(X, \mathcal{L}, \sigma)$ is the twisted homogeneous coordinate ring defined in* [AV90]*. If A is noetherian then ϕ is surjective in large degree.*

The kernel of ϕ is equal in large degree to

$$J = \bigcap \{\operatorname{Ann}_A(M) \mid M \text{ is a } C\text{-point module for some commutative } \Bbbk\text{-algebra } C\}.$$

The purpose of this section is to give a version of this theorem for a (not necessarily connected graded) algebra graded by an arbitrary monoid Γ.

We first need some notation. Let Γ be a monoid and let A be a Γ-graded ring. If M is a Γ-graded right A-module and $\gamma \in \Gamma$, we define the *shift* $M(\gamma)$ of M by γ as:

$$M(\gamma) = \bigoplus_{\delta \in \Gamma} M(\gamma)_\delta,$$

where $M(\gamma)_\delta = M_{\gamma\delta}$. We note that

$$M(\gamma)_\delta A_\epsilon = M_{\gamma\delta} A_\epsilon \subseteq M_{\gamma\delta\epsilon} = M(\gamma)_{\delta\epsilon}, \tag{2.1}$$

so $M(\gamma)$ is again a Γ-graded right A-module. Note that

$$(M(\gamma))(\delta)_\epsilon = M(\gamma)_{\delta\epsilon} = M_{\gamma\delta\epsilon} = M(\gamma\delta)_\epsilon$$

and so $(M(\gamma))(\delta)$ is canonically isomorphic to $M(\gamma\delta)$.

If M is a left module we define $M(\gamma)_\delta = M_{\delta\gamma}$. Then (2.1) becomes:

$$A_\epsilon M(\gamma)_\delta = A_\epsilon M_{\delta\gamma} \subseteq M_{\epsilon\delta\gamma} = M(\gamma)_{\epsilon\delta},$$

as needed. We have

$$(M(\gamma))(\delta)_\epsilon = M(\gamma)_{\epsilon\delta} = M_{\epsilon\delta\gamma} = M(\delta\gamma)_\epsilon$$

so $(M(\gamma))(\delta)$ is canonically isomorphic to $M(\delta\gamma)$.

If A is a Γ-graded ring, an *intermediate series* module over A is a Γ-graded left or right A-module M so that $\dim M_\gamma = 1$ for all $\gamma \in \Gamma$. We will use a shift-invariant family of intermediate series modules to construct a ring homomorphism from A to a Γ-graded ring, giving a version of Theorem 2.1 in this setting.

Our notation for smash products is that if Γ acts on A then $A \rtimes \Gamma = \bigoplus_{\gamma \in \Gamma} At^\gamma$, where $t^\gamma t^\delta = t^{\gamma\delta}$ and $t^\gamma r = r^\gamma t^\gamma$ for all $r \in A$, $\gamma \in \Gamma$.

Theorem 2.2. *Let Γ be a monoid with identity e and let A be a Γ-graded ring. Let X be a reduced affine scheme that parameterises a set of intermediate series right A-modules, in the sense that for $x \in X$ there is a module M^x with basis $\{v^x_\gamma \mid \gamma \in \Gamma\}$, and that there is a $\Bbbk$-linear function $\phi : A \to \Bbbk[X]$ so that*

$$v^x_e r = \phi(r)(x) v^x_\gamma$$

for all $\gamma \in \Gamma, r \in A_\gamma$. Further suppose that shifting defines a group antihomomorphism $\sigma : \Gamma \to \operatorname{Aut}_\Bbbk(X), \gamma \mapsto \sigma^\gamma$ so that $M^x(\gamma) \cong M^{\sigma^\gamma(x)}$. Here we require that the isomorphism maps $v^x_{\gamma\delta} \mapsto v^{\sigma^\gamma(x)}_\delta$.

In this setting the map

$$\Phi : A \to \Bbbk[X] \rtimes \Gamma, \quad r \in A_\gamma \mapsto \phi(r)t^\gamma$$

is a graded homomorphism of algebras. Further,

$$\ker \Phi = \bigcap_{x \in X} \operatorname{Ann}_A M^x.$$

Proof. Let Γ act on $\Bbbk[X]$ by $f^\gamma = (\sigma^\gamma)^*(f)$, so σ defines a homomorphism from $\Gamma \to \operatorname{Aut}_\Bbbk(\Bbbk[X])$.

Let $r \in A_\gamma, s \in A_\delta$, and let $\alpha : V^x(\gamma) \to V^{\sigma^\gamma(x)}$ be the given isomorphism. Then:

$$\alpha(v^x_\gamma s) = v^{\sigma^\gamma(x)}_e s = \phi(s)(\sigma^\gamma(x)) v^{\sigma^\gamma(x)}_\delta = \alpha(\phi(s)(\sigma^\gamma(x)) v^x_{\gamma\delta}).$$

So

$$v^x_\gamma s = \phi(s)^\gamma(x) v^x_{\gamma\delta}. \tag{2.2}$$

Now, using (2.2), we obtain:

$$\phi(rs)(x) v^x_{\gamma\delta} = v^x_e rs = \phi(r)(x) v^x_\gamma s = \phi(r)(x)\phi(s)^\gamma(x) v^x_{\gamma\delta}$$

and so

$$\phi(rs) = \phi(r)\phi(s)^\gamma. \tag{2.3}$$

Then by (2.3) we have

$$\Phi(rs) = \phi(rs)t^{\gamma\delta} = \phi(r)\phi(s)^\gamma t^{\gamma\delta} = \phi(r)t^\gamma\phi(s)t^\delta = \Phi(r)\Phi(s).$$

Since Φ is graded, $\ker\Phi$ is a graded ideal of A. If $r \in A$ is homogeneous then

$$\Phi(r) = 0 \iff \phi(r) = 0 \iff v_e^x r = 0 \text{ for all } x \in X.$$

Let $\gamma \in \Gamma$. Then

$$v_e^x r = 0 \text{ for all } x \in X \iff v_e^{\sigma^\gamma(x)} r = 0 \text{ for all } x \in X \iff v_\gamma^x r = 0 \text{ for all } x \in X,$$

using the isomorphism between $M^x(\gamma)$ and $M^{\sigma^\gamma(x)}$. So

$$\Phi(r) = 0 \iff v_\gamma^x r = 0 \text{ for all } x \in X, \gamma \in \Gamma \iff r \in \bigcap_{x \in X} \operatorname{Ann}_A M^x.$$

□

(The reason we require X in the theorem statement to be reduced is that we are constructing Φ from the closed points of X, and so effectively from the reduced induced structure on X.)

Remark 2.3. We need the map σ in Theorem 2.2 to be an antihomomorphism because of the equations:

$$M^{\sigma^{\gamma\delta}(x)} \cong M^x(\gamma\delta) = (M^x(\gamma))(\delta) \cong M^{\sigma^\gamma(x)}(\delta) \cong M^{\sigma^\delta(\sigma^\gamma(x))}.$$

Remark 2.4. There is a universal module M for the family $\{M^x \mid x \in X\}$, which is isomorphic as a $\Bbbk[X]$-module to $\bigoplus_{\gamma\in\Gamma} \Bbbk[X]v_\gamma$. The module structure is given by

$$v_\gamma s = \phi(s)^\gamma v_{\gamma\delta} \tag{2.4}$$

for $s \in A_\delta$. If we consider the natural right action of A on $M = \Bbbk[X] \rtimes \Gamma$ then we have $t^\gamma \cdot s = t^\gamma\Phi(s) = t^\gamma\phi(s)t^\delta = \phi(s)^\gamma t^{\gamma\delta}$ for $s \in A_\delta$. This agrees with (2.4) if we set $v_\gamma = t^\gamma$, and so $M \cong \Bbbk[X] \rtimes \Gamma$.

The theorem for left modules is:

Theorem 2.5. *Let Γ be a monoid with identity e and let A be a Γ-graded ring. Let X be a reduced affine scheme that parameterises a set of intermediate series left A-modules, in the sense that the left module N^x has a basis $\{v_\gamma^x \mid \gamma \in \Gamma\}$ and that there is a $\Bbbk$-linear function $\phi : A \to \Bbbk[X]$ so that*

$$rv_e^x = \phi(r)(x)v_\gamma^x$$

for all $\gamma \in \Gamma, r \in A_\gamma$. Further suppose that shifting defines a group homomorphism $\sigma : \Gamma \to \operatorname{Aut}_\Bbbk(X), \gamma \mapsto \sigma^\gamma$ so that $N^x(\gamma) \cong N^{\sigma^\gamma(x)}$. Here we require that the isomorphism maps $v_{\delta\gamma}^x \mapsto v_\delta^{\sigma^\gamma(x)}$.

In this setting the map

$$\Phi : A \to \Bbbk[X] \rtimes \Gamma^{op} \quad r \in A_\gamma \mapsto \phi(r)t^\gamma$$

is a graded antihomomorphism of algebras. Further,

$$\ker\Phi = \bigcap_{x\in X} \operatorname{Ann}_A N^x.$$

Proof. We repeat the proof above to ensure that the change of notation from right to left is handled correctly. Again, let $f^\gamma = (\sigma^\gamma)^* f$, so σ defines a homomorphism from $\Gamma^{op} \to \operatorname{Aut}_\Bbbk \Bbbk[X]$. Let $r \in A_\gamma, s \in A_\delta$, and let $\alpha : V^x(\delta) \to V^{\sigma^\delta(x)}$ be the given isomorphism. Then:

$$\alpha(rv_\delta^x) = rv_e^{\sigma^\delta(x)} = \phi(r)(\sigma^\delta(x))v_\gamma^{\sigma^\delta(x)} = \alpha(\phi(r)(\sigma^\delta(x))v_{\gamma\delta}^x).$$

So

$$rv_\delta^x = \phi(r)(\sigma^\delta(x))v_{\gamma\delta}^x. \tag{2.5}$$

Now, using (2.5), we obtain:

$$\phi(rs)(x)v_{\gamma\delta}^x = rsv_e^x = \phi(s)(x)rv_\delta^x = \phi(s)(x)\phi(r)(\sigma^\delta(x))v_{\gamma\delta}^x$$

and so

$$\phi(rs) = \phi(s)\phi(r)^\delta. \tag{2.6}$$

Then by (2.6) we have

$$\Phi(rs) = \phi(s)\phi(r)^\delta t^{\gamma\delta} = \phi(s)\phi(r)^\delta t^{\delta\circ_{\mathrm{op}}\gamma} = \phi(s)t^\delta\phi(r)t^\gamma = \Phi(s)\Phi(r).$$

The proof of the last statement is identical to the proof in Theorem 2.2. □

Remark 2.6. We need the map σ in Theorem 2.5 to be a homomorphism because:

$$N^{\sigma^{\gamma\delta}(x)} = N^x(\gamma\delta) = (N^x(\delta))(\gamma) = N^{\sigma^\delta(x)}(\gamma) = N^{\sigma^\gamma(\sigma^\delta(x))}.$$

Note also that a graded anti-homomorphism from a Γ-graded algebra should map to a Γ^{op}-graded algebra, as we indeed have.

Remark 2.7. We likewise obtain the universal left module for the N^x from Φ. Set $N = \Bbbk[X] \rtimes \Gamma^{op}$. The left action induced by Φ is $r \cdot \delta = \delta\Phi(r)$ because Φ is an anti-homomorphism, so we get

$$r \cdot t^\delta = t^\delta\Phi(r) = t^\delta\phi(r)t^\gamma = \phi(r)^\delta t^{\delta\circ_{op}\gamma} = \phi(r)^\delta t^{\gamma\delta}$$

for $r \in A_\gamma$, which is the structure we expect.

Remark 2.8. Let $\mathrm{Bir}(X)$ be the group of birational self-maps of X. In the settings above, suppose that shifting defines elements of $\mathrm{Bir}(X)$, in the sense that σ maps Γ to $\mathrm{Bir}(X)$. We get a generalization of Theorems 2.2 and 2.5 by replacing $\Bbbk[X]$ and $\mathrm{Aut}(\Bbbk[X])$ with $\Bbbk(X)$ and $\mathrm{Bir}(X)$, respectively.

3. Intermediate series modules over higher rank Witt algebras

Let Γ be a rank n $\mathbb{Z}$-submodule of $\Bbbk$. The *rank n Witt algebra* W_Γ (or *higher rank Witt algebra* if $n \geq 2$, sometimes called the centerless higher rank Virasoro algebra) is the Lie algebra with $\Bbbk$-basis $\{e_\nu \mid \nu \in \Gamma\}$ and bracket

$$[e_\mu, e_\nu] = (\nu - \mu)e_{\nu+\mu}$$

for $\nu, \mu \in \Gamma$. The rank one Witt algebra is the "usual" Witt algebra, which we denote by W.

As $U(W_\Gamma)$ is Γ-graded one can consider intermediate series modules as in Section 2. They are the standard intermediate series modules of Lie algebras, called also Harish-Chandra modules over $(W_\Gamma, \Bbbk e_0)$; i.e., modules of the form $\oplus_{\gamma\in\Gamma}V_\gamma$, where V_γ is the γ-eigenspace for e_0 and has dimension 1.

The intermediate series W_Γ-modules have been classified in [Su94, Theorem 2.1], generalizing the classification [KS85] for the Witt algebra. There are three families of indecomposable intermediate series W_Γ-modules:

$$V_{(\alpha,\beta)} = \oplus_{\nu\in\Gamma}\Bbbk v_\nu, \quad e_\mu v_\nu = (\alpha + \beta\mu + \nu)v_{\mu+\nu},$$

$$A_{(\alpha,\beta)} = \oplus_{\nu\in\Gamma}\Bbbk v_\nu, \quad e_\mu v_\nu = \begin{cases} \nu v_{\mu+\nu} & \nu \neq 0,\ \mu+\nu \neq 0, \\ (\alpha+\beta\mu)v_\mu & \nu = 0, \\ 0 & \mu+\nu = 0, \end{cases}$$

$$B_{(\alpha,\beta)} = \oplus_{\nu\in\Gamma}\Bbbk v_\nu, \quad e_\mu v_\nu = \begin{cases} (\mu+\nu) v_{\mu+\nu} & \nu \neq 0,\ \mu+\nu \neq 0, \\ 0 & \nu = 0, \\ (\alpha+\beta\mu)v_0 & \mu+\nu = 0, \end{cases}$$

where $(\alpha, \beta) \in \mathbb{A}^2$. Note that $A_{(\alpha,\beta)}$, $B_{(\alpha,\beta)}$ are only defined where $(\alpha, \beta) \neq (0, 0)$ and depend up to isomorphism (rescaling of v_0) only on $[\alpha : \beta] \in \mathbb{P}^1$. We will therefore denote them by $A_{[\alpha:\beta]}$, $B_{[\alpha:\beta]}$. Note also that we have $A_{[1:0]} \cong V_{(0,1)}$ (by $v_0 \mapsto v_0$ and $v_\nu \mapsto \nu v_\nu$ when $\nu \neq 0$) and $B_{[1:0]} \cong V_{(0,0)}$ (by $v_0 \mapsto \nu v_0$ and $v_\nu \mapsto v_\nu$ when $\nu \neq 0$).

Remark 3.1. Note that $A_{[\alpha:\beta]}$ contains a simple submodule $\oplus_{0\neq\nu\in\Gamma}\Bbbk v_\nu$ with a 1-dimensional trivial quotient. On the other hand, $B_{[\alpha:\beta]}$ has the 1-dimensional trivial submodule $\Bbbk v_\nu$, and the quotient is a simple module. This is explained by the isomorphism $B'_{[\alpha:\beta]} \cong A_{[\alpha:\beta]}$, where $'$ denotes the adjoint. (If $M = \oplus_{\gamma\in\Gamma}\Bbbk v_\gamma$ is a left Γ-graded W_Γ-module, the *adjoint* (or *restricted dual*) of M is the left Γ-graded W_Γ-module M' with $M'_\gamma = \mathrm{Hom}_\Bbbk(M_{-\gamma}, \Bbbk)$, $v'_\gamma = v^*_{-\gamma}$, and $e_\mu v'_\gamma = -v^*_{-\gamma}e_\mu$.)

Remark 3.2. We use a slightly different presentation of the families $A_{[\alpha:\beta]}$, $B_{[\alpha:\beta]}$ than in [Su94]. In loc.cit the last two families are replaced by $\tilde{A}(a')$ defined by

$$e_\mu v'_\nu = (\nu+\mu)v'_{\mu+\nu},\ \nu \neq 0, \quad e_\mu v_0 = \mu(1+(\mu+1)a')v'_\mu,$$

and by $\tilde{B}(a')$ defined by

$$e_\mu v'_\nu = \nu v'_{\mu+\nu},\ \nu \neq -\mu, \quad e_\mu v'_{-\mu} = -\mu(1+(\mu+1)a')v'_0,$$

for $a' \in \Bbbk \cup \{\infty\}$. If $a' = \infty$ then $1+(\mu+1)a'$ in the above definition is regarded as $\mu+1$. Note that $\tilde{A}(a')$ (resp. $\tilde{B}(a')$) is isomorphic to $A_{[1+a':a']}$ (resp. $B_{[1+a':a']}$) if $a' \neq \infty$ and to $A_{[1:1]}$ (resp. $B_{[1:1]}$) if $a' = \infty$, for $v_\nu = \nu v'_\nu$ (resp. $v_\nu = \frac{1}{\nu}v'_\nu$) if $\nu \neq 0$, and $v_0 = v'_0$.

For the Witt algebra the choice of the basis is the same in [KS85], however there $a' \in \Bbbk$ and modules are classified up to inversion: replacing v_ν by $-v_{-\nu}$.

Let us show how to obtain the intermediate series modules using results of Section 2.

Proposition 3.3. *Let Γ act on $\Bbbk[a,b]$ as $t^\nu.p(a,b) = p(a+\nu,b)t^\nu$, and let $T := \Bbbk[a,b] \rtimes \Gamma$. The map $\phi : W_\Gamma \to T$, $\phi(e_\mu) = (a+b\mu)t^\mu$, induces an anti-homomorphism $\Phi : U(W_\Gamma) \to T$. Consequently, T is a left $U(W)$-module via $e_\mu.p(a,b)t^\nu = (a+\nu+b\mu)p(a,b)t^{\mu+\nu}$.*

Proof. Note that $\mathbb{A}^2$ parametrises a set of intermediate series modules $N^{(\alpha,\beta)} := V_{(\alpha,\beta)}$ and $e_\mu v_0^{(\alpha,\beta)} = (a+b\mu)((\alpha,\beta))v_\mu^{(\alpha,\beta)}$. Further, $N^{(\alpha,\beta)}(\nu) \cong N^{(\alpha+\nu,\beta)}$ and hence $\sigma^\nu((\alpha,\beta)) = (\alpha+\nu,\beta)$ (using the notation of Section 2). The proposition therefore follows by Theorem 2.5 and Remark 2.7. □

Remark 3.4. Let $\Gamma = \mathbb{Z}$ and $T = \Bbbk[a,b] \rtimes \mathbb{Z}$. We may compose the map Φ of Proposition 3.3 with the canonical anti-automorphism $e_n \mapsto -e_n$ of $U(W)$ to obtain a homomorphism $\Phi' : U(W) \to T, e_n \mapsto (-a-bn)t^n$.

Recall that in [SW15] a homomorphism $\hat{\phi}$ was constructed from $U(W)$ to

$$T' := \Bbbk\langle u, v, v^{-1}, w\rangle/(uv - vu - v^2, uw - wu - wv, vw - wv),$$

defined by $\hat{\phi}(e_n) = (u-(n-1)w)v^{n-1}$. The reader may verify that $\alpha : T' \to T$ defined by

$$u \mapsto (b-a)t, \quad v \mapsto t, \quad w \mapsto bt$$

is an isomorphism of graded rings and that $\alpha\hat{\phi} = \Phi'$. Thus Proposition 3.3 generalises the construction of $\hat{\phi}$.

We now discuss applications of Φ to the representation theory of W_Γ. For $p = (\alpha,\beta) \in \mathbb{A}^2$ we denote by $I(p)$ the ideal $(a-\alpha, b-\beta)$ in $\Bbbk[a,b]$. For q infinitely near to p, corresponding to $[x:y] \in \mathbb{P}^1$, we denote by $I(q)$ the ideal $(y(a-\alpha)-x(b-\beta), (a-\alpha)^2, (a-\alpha)(b-\beta), (b-\beta)^2)$.

Let $B = \Phi(U(W_\Gamma))$, and note that B is contained in the double idealizer $R = \Bbbk[a,b] + (I(0,0)T \cap TI(0,1))$. From the discussion in the introduction, then, we expect three families of intermediate series $U(W_\Gamma)$-modules, one parameterised by $\mathbb{A}^2 \smallsetminus \{(0,0),(0,1)\}$ and two parameterised by $\mathbb{P}^1$. Note that because Φ is an anti-homomorphism, *right* B-modules will correspond to *left* $U(W_\Gamma)$-modules.

By construction of Φ we have $V(\alpha,\beta) \cong T/I(p)T$, considered as a B-module. Removing $V(0,0)$ and $V(0,1)$ we obtain the two-dimensional family we expect. We next show that we also obtain the two $\mathbb{P}^1$-families $A_{[\alpha:\beta]}$ and $B_{[\alpha:\beta]}$.

Proposition 3.5. *Let $[x:y] \in \mathbb{P}^1$ and let $I(q) = (ya-xb, a^2, ab, b^2)$ define a point infinitely near to $(0,0)$. Let*

$$P(q) = \frac{\Bbbk[a,b] + I(0,0)T}{I(0,0) + I(q)T}.$$

Then $A_{[x:y]} \cong P(q)$.

Proof. If $w \in \Bbbk[a,b] + I(0,0)T$ let $\overline{w}$ be the image of w in $P(q)$. If $x \neq 0$ we choose a basis

$$v_\nu = \begin{cases} \overline{at^\nu} & \nu \neq 0, \\ \overline{1} & \nu = 0 \end{cases}$$

for $P(q)$.

Using the anti-homomorphism, we compute for $\nu \neq 0$

$$e_\mu.v_\nu = \overline{at^\nu(a+b\mu)t^\mu} = \overline{a(a+b\mu+\nu)t^{\mu+\nu}} = \nu\overline{at^{\mu+\nu}} = \begin{cases} \nu v_{\nu+\mu} & \nu+\mu \neq 0, \\ 0 & \nu+\mu = 0. \end{cases}$$

and

$$e_\mu.v_0 = \overline{(a+b\mu)t^\mu} = \overline{\left(a+\frac{y}{x}a\mu\right)t^\mu} = \left(1+\frac{y}{x}\mu\right)v_\mu,$$

so $P(q) \cong A_{[x:y]}$ as claimed.

If $y \neq 0$ we pick a basis

$$v_\nu = \begin{cases} \overline{bt^\nu} & \nu \neq 0, \\ \overline{1} & \nu = 0, \end{cases}$$

and obtain $e_\mu.v_\nu = \nu v_{\nu+\mu}$, $e_\mu.v_0 = (\frac{x}{y}+\mu)v_\mu$, $e_\mu.v_{-\mu} = 0$. Thus $P(q) \cong A_{[x:y]}$ again. □

In the next result, note the change of sides from the left modules $Q(q)$ defined in the introduction.

Proposition 3.6. *Let $[x:y] \in \mathbb{P}^1$ and let $I(q) = (ya - x(b-1), a^2, a(b-1), (b-1)^2)$ define a point infinitely near to $(0,1)$. Let*

$$Q(q) = \frac{I(0,1) \oplus \bigoplus_{0\neq\nu\in\Gamma} \Bbbk[a,b]t^\nu}{I(q) \oplus \bigoplus_{0\neq\nu\in\Gamma} I(0,1)t^\nu}.$$

Then $B_{[x:y]} \cong Q(q)$.

Proof. If $x \neq 0$ we choose a basis

$$v_\nu = \begin{cases} \overline{t^\nu} & \nu \neq 0, \\ \overline{a} & \nu = 0 \end{cases}$$

for $Q(q)$. We compute for $\nu+\mu \neq 0$, $\nu \neq 0$

$$e_\mu.v_\nu = \overline{(a+b\mu+\nu)t^{\mu+\nu}} = (\mu+\nu)\overline{t^{\mu+\nu}} = (\mu+\nu)v_{\mu+\nu}$$

and

$$e_\mu.v_0 = \overline{a(a+b\mu)t^\mu} = 0, \quad e_\mu.v_{-\mu} = \overline{a+b\mu-\mu} = \left(1+\frac{y}{x}\mu\right)v_0.$$

If $y \neq 0$ we pick a basis

$$v_\nu = \begin{cases} \nu\overline{t^\nu} & \nu \neq 0, \\ \overline{b} & \nu = 0. \end{cases}$$

We get $e_\mu.v_\nu = \nu v_{\mu+\nu}$, $e_\mu.v_0 = 0$, $e_\mu.v_{-\mu} = \left(\frac{x}{y}+\mu\right)v_0$. □

4. Factors of $U(W_\Gamma)$

In this section we generalise techniques from [SW15] to show that $B = \Phi(U(W_\Gamma))$ is not left or right noetherian. This in particular implies that $U(W_\Gamma)$ is not left or right noetherian, which was proved earlier in [SW14, SW15].

For $0 \neq \mu \in \Gamma$, let

$$p_\mu = e_\mu e_{3\mu} - e_{2\mu}^2 - \mu e_{4\mu}.$$

Lemma 4.1. *We have $\Phi(p_\mu) = \mu^2 b(1-b)t^{4\mu}$.*

Proof. Let us compute

$$\begin{aligned}\Phi(e_\mu e_{3\mu} - e_{2\mu}^2 - \mu e_{4\mu}) &= \left((a+3\mu b)(a+\mu b+3\mu) - (a+2\mu b)(a+2\mu b+2\mu) - \mu(a+4\mu b)\right)t^{4\mu} \\ &= \mu^2 b(1-b)t^{4\mu}.\end{aligned}$$

□

Fix $0 \neq \mu \in \Gamma$ and let $I = B\Phi(p_\mu)B$.

Lemma 4.2. *For all $\nu \in \Gamma$ we have $b(1-b)t^\nu \in I$. In particular, I does not depend on the choice of μ. Consequently, $I = b(1-b)\Bbbk[a,b] \rtimes \Gamma$.*

Proof. We have

$$\Phi(e_{\nu-4\mu})b(1-b)t^{4\mu} - b(1-b)t^{4\mu}\Phi(e_{\nu-4\mu}) = (\Phi(e_{\nu-4\mu}) - \Phi(e_{\nu-4\mu}) - 4\mu)b(1-b)t^{\nu} = -4\mu b(1-b)t^{\nu}.$$

Thus the first claim follows by Lemma 4.1. Note that $I \subseteq b(1-b)\Bbbk[a,b] \rtimes \Gamma$, and as $b(1-b) \in I$ and $a \in B$, we have $b(1-b)\Bbbk[a] \rtimes \Gamma \subseteq I$. Since also $(a+b\mu)t^{\mu} \in B$, we easily obtain by induction on n that $b(1-b)b^n\Bbbk[a] \rtimes \Gamma \subseteq I$ for all $n \geq 0$, and thus the last claim. □

Proposition 4.3. *The ideal I is not finitely generated as a left or right ideal of B.*

Proof. We first compute

$$(a+b\nu_1)t^{\nu_1}\cdots(a+b\nu_l)t^{\nu_l}p(a,b)b(1-b)t^{\lambda} = (a+b\nu_1)\cdots(a+b\nu_l+\nu_1+\cdots+\nu_{l-1})p(a+\nu_1+\cdots+\nu_{l-1}+\nu_l,b)b(1-b)t^{\nu_1+\cdots+\nu_l+\lambda}, \tag{4.1}$$

$$p(a,b)b(1-b)t^{\lambda}(a+b\nu_1)t^{\nu_1}\cdots(a+b\nu_l)t^{\nu_l} = p(a,b)b(1-b)(a+b\nu_1+\lambda)\cdots(a+b\nu_l+\lambda+\nu_1+\cdots+\nu_{l-1})t^{\lambda+\nu_1+\cdots+\nu_l}. \tag{4.2}$$

Let us assume that I is finitely generated as a left ideal of B. Then there exist $\mu_1,\ldots,\mu_k \in \Gamma$ such that $I = B(I_{\mu_1}+\cdots+I_{\mu_k})$. Let us take $\mu \neq \mu_i$, $1 \leq i \leq k$. It follows from (4.1) that $(B(I_{\mu_1}+\cdots+I_{\mu_k}))_{\mu}$ is contained in $(a,b)b(1-b)t^{\mu}$, a contradiction to Lemma 4.2.

Let us assume now that I is finitely generated as a right ideal in B. Then there exist $\mu_1,\ldots,\mu_k \in \Gamma$ such that $I = (I_{\mu_1}+\cdots+I_{\mu_k})B$. For $\mu \neq \mu_i$, $1 \leq i \leq k$, we obtain from (4.2) that $((I_{\mu_1}+\cdots+I_{\mu_k})B)_{\mu}$ is contained in $(a+\mu,b-1)b(1-b)t^{\mu}$, which again contradicts Lemma 4.2. □

Remark 4.4. Note that the same proof works if Γ is a submonoid of $\Bbbk$. Lemma 4.2 yields in this case $b(1-b)t^{n\mu} \in I$, for $n \geq 4$. The proof of Proposition 4.3 can then be adapted in an obvious way to apply to this a slightly more general situation. In particular, $\Phi(U(W_+))$ is not noetherian, where W_+ is the subalgebra of W generated by $\{e_n : n \geq 1\}$. (This last statement is proved in [SW15]

We now show that the image B_β of the map $\phi_\beta : U(W) \to B/(b-\beta)$ induced from Φ is noetherian for every $\beta \in \Bbbk$. This is an analogue of [SW15, Proposition 2.1].

Lemma 4.5. *We have $B_0 \cong \Bbbk + a(\Bbbk[a] \rtimes \Gamma)$, $B_1 \cong \Bbbk + (\Bbbk[a] \rtimes \Gamma)a$, $B_\beta \cong \Bbbk[a] \rtimes \Gamma$ for $\beta \neq 0,1$.*

Proof. The lemma is obvious for $\beta = 0,1$. Assume therefore that $\beta \neq 0,1$. Let us compute

$$(a+\beta\mu)t^{\mu}(a+\beta\nu)t^{\nu} - a(a+\beta(\mu+\nu))t^{\mu+\nu} = (\mu a + \beta\mu(\beta\nu+\mu))t^{\mu+\nu} \in B_\beta.$$

Subtracting $\mu(a+b(\mu+\nu)t^{\mu+\nu}$, we thus have $\beta\mu\nu(\beta-1)t^{\mu+\nu} \in B_\beta$, and hence our claim. □

Proposition 4.6. *B_β is noetherian for every $\beta \in \Bbbk$.*

Proof. For $\beta \neq 0,1$ this follows by [MR01, Theorem 4.5] using Lemma 4.5. Let us note that $B_0 \cong B_1$ by conjugation with a. It thus suffices to prove that B_0 is right noetherian and B_1 is left noetherian. We show that B_0 is right noetherian, and following the same argument one can show that B_1 is left noetherian.

We first note that $I = a(\Bbbk[a] \rtimes \Gamma)$ is a maximal right ideal in $C = \Bbbk[a] \rtimes \Gamma$. To see this, let $J \neq I$ be a right ideal which contains I. Take an element $c = \sum \alpha_\mu t^\mu \neq 0$ in J with the minimal number of nonzero coefficients. Since $ca = \sum \alpha_\mu(a+\mu)t^\mu \in J$ and hence $\sum \alpha_\mu \mu t^\mu \in J$, the minimality assumption implies that $J = \Bbbk[a] \rtimes \Gamma$.

The proposition now follows by [Rob72, Theorem 2.2] using Lemma 4.5. □

Remark 4.7. We remark that for any β the modules $V(\alpha,\beta)$ are all faithful over B_β, and it follows easily that the B_β are primitive. In general, the primitive factors of $U(W_\Gamma)$ are unknown, even for $\Gamma = \mathbb{Z}$.

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